CHAPTER 6

Interpolation Functions for General Element Formulation

6.1 INTRODUCTION

The structural elements introduced in the previous chapters were formulated on the basis of known principles from elementary strength of materials theory. We have also shown, by example, how Galerkin's method can be applied to a heat conduction problem. This chapter examines the requirements for interpolation functions in terms of solution accuracy and convergence of a finite element analysis to the exact solution of a general field problem. Interpolation functions for various common element shapes in one, two, and three dimensions are developed, and these functions are used to formulate finite element equations for various types of physical problems in the remainder of the text.

With the exception of the beam element, all the interpolation functions discussed in this chapter are applicable to finite elements used to obtain solutions to problems that are said to be C^0-continuous. This terminology means that, across element boundaries, only the zeroth-order derivatives of the field variable (i.e., the field variable itself) are continuous. On the other hand, the beam element formulation is such that the element exhibits C^1-continuity, since the first derivative of the transverse displacement (i.e., slope) is continuous across element boundaries, as discussed previously and repeated later for emphasis. In general, in a problem having C^n-continuity, derivatives of the field variable up to and including nth-order derivatives are continuous across element boundaries.

6.2 COMPATIBILITY AND COMPLETENESS REQUIREMENTS

The line elements (spring, truss, beam) illustrate the general procedures used to formulate and solve a finite element problem and are quite useful in analyzing truss and frame structures. Such structures, however, tend to be well defined in terms of the number and type of elements used. In most engineering problems, the domain of interest is a continuous solid body, often of irregular shape, in which the behavior of one or more field variables is governed by one or more partial differential equations. The objective of the finite element method is to discretize the domain into a number of finite elements for which the governing equations are algebraic equations. Solution of the resulting system of algebraic equations then gives an *approximate* solution to the problem. As with any approximate technique, the question, How accurate is the solution? must be addressed.

In finite element analysis, solution accuracy is judged in terms of convergence as the element "mesh" is refined. There are two major methods of mesh refinement. In the first, known as *h-refinement,* mesh refinement refers to the process of increasing the number of elements used to model a given domain, consequently, reducing individual element size. In the second method, *p-refinement,* element size is unchanged but the order of the polynomials used as interpolation functions is increased. The objective of mesh refinement in either method is to obtain sequential solutions that exhibit asymptotic convergence to values representing the exact solution. While the theory is beyond the scope of this book, mathematical proofs of convergence of finite element solutions to correct solutions are based on a specific, regular mesh refinement procedure defined in [1]. Although the proofs are based on regular meshes of elements, irregular or unstructured meshes (such as in Figure 1.7) can give very good results. In fact, use of unstructured meshes is more often the case, since (1) the geometries being modeled are most often irregular and (2) the automeshing features of most finite element software packages produce irregular meshes.

An example illustrating regular *h*-refinement as well as solution convergence is shown in Figure 6.1a, which depicts a rectangular elastic plate of uniform thickness fixed on one edge and subjected to a concentrated load on one corner.

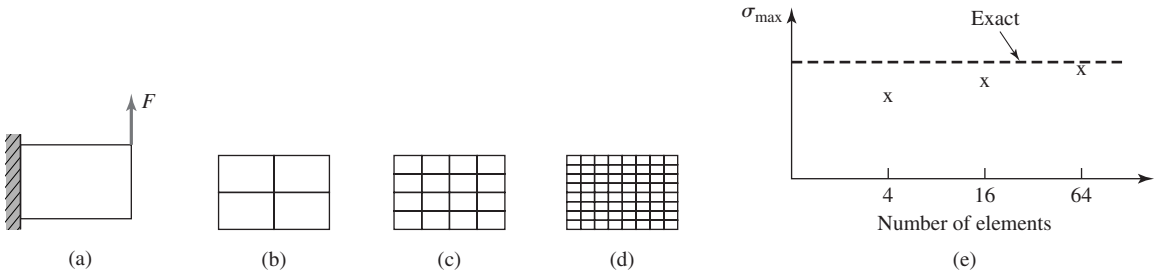

Figure 6.1 Example showing convergence as element mesh is refined.

This problem is modeled using rectangular plane stress elements (Chapter 9) and three meshes used in sequence, as shown (Figure 6.1b–6.1d). Solution convergence is depicted in Figure 6.1e in terms of maximum normal stress in the x direction. For this example, the exact solution is taken to be the maximum bending stress computed using elementary beam theory. The true exact solution is the plane stress solution from the theory of elasticity. However, the maximum normal stress is not appreciably changed in the elasticity solution.

The need for convergence during regular mesh refinement is rather clear. If convergence is not obtained, the engineer using the finite element method has absolutely no indication whether the results are indicative of a meaningful approximation to the correct solution. For a general field problem in which the field variable of interest is expressed on an element basis in the discretized form

$$\phi^{(e)}(x, y, z) = \sum_{i=1}^{M} N_i(x, y, z)\phi_i \tag{6.1}$$

where M is the number of element degrees of freedom, the interpolation functions must satisfy two primary conditions to ensure convergence during mesh refinement: the *compatibility* and *completeness* requirements, described as follows.

6.2.1 Compatibility

Along element boundaries, the field variable and its partial derivatives up to one order less than the highest-order derivative appearing in the integral formulation of the element equations must be continuous. Given the discretized representation of Equation 6.1, it follows that the interpolation functions must meet this condition, since these functions determine the spatial variation of the field variable.

Recalling the application of Galerkin's method to the formulation of the truss element equations, the first derivative of the displacement appears in Equation 5.34. Therefore, the displacement must be continuous across element boundaries, but none of the displacement derivatives is required to be continuous across such boundaries. Indeed, as observed previously, the truss element is a constant strain element, so the first derivative is, in general, discontinuous at the boundaries. Similarly, the beam element formulation, Equation 5.49, includes the second derivative of displacement, and compatibility requires continuity of both the displacement and the slope (first derivative) at the element boundaries.

In addition to satisfying the criteria for convergence, the compatibility condition can be given a physical meaning as well. In structural problems, the requirement of displacement continuity along element boundaries ensures that no gaps or voids develop in the structure as a result of modeling procedure. Similarly the requirement of slope continuity for the beam element ensures that no "kinks" are developed in the deformed structure. In heat transfer problems, the compatibility requirement prevents the physically unacceptable possibility of jump discontinuities in temperature distribution.

6.2.2 Completeness

In the limit as element size shrinks to zero in mesh refinement, the field variable and its partial derivatives up to, and including, the highest-order derivative appearing in the integral formulation must be capable of assuming constant values. Again, because of the discretization, the completeness requirement is directly applicable to the interpolation functions.

The completeness requirement ensures that a displacement field within a structural element can take on a constant value, representing rigid body motion, for example. Similarly, constant slope of a beam element represents rigid body rotation, while a state of constant temperature in a thermal element corresponds to no heat flux through the element. In addition to the rigid body motion consideration, the completeness requirement also ensures the possibility of constant values of (at least) first derivatives. This feature assures that a finite element is capable of constant strain, constant heat flow, or constant fluid velocity, for example.

The foregoing discussion of convergence and requirements for interpolation functions is by no means rigorous nor greatly detailed. References [1–5] lead the interested reader to an in-depth study of the theoretical details. The purpose here is to present the requirements and demonstrate application of those requirements to development of appropriate interpolation functions to a number of commonly used elements of various shape and complexity.

6.3 POLYNOMIAL FORMS: ONE-DIMENSIONAL ELEMENTS

As illustrated by the methods and examples of Chapter 5, formulation of finite element characteristics requires differentiation and integration of the interpolation functions in various forms. Owing to the simplicity with which polynomial functions can be differentiated and integrated, polynomials are the most commonly used interpolation functions. Recalling the truss element development of Chapter 2, the displacement field is expressed via the first-degree polynomial

$$u(x) = a_0 + a_1 x \tag{6.2}$$

In terms of nodal displacement, Equation 6.2 is determined to be equivalent to

$$u(x) = \left(1 - \frac{x}{L}\right)u_1 + \frac{x}{L}u_2 \tag{6.3}$$

The coefficients a_0 and a_1 are obtained by applying the nodal conditions $u(x = 0) = u_1$ and $u(x = L) = u_2$. Then, collecting coefficients of the nodal displacements, the interpolation functions are obtained as

$$N_1 = 1 - \frac{x}{L} \qquad N_2 = \frac{x}{L} \tag{6.4}$$

Equation 6.3 shows that, if $u_1 = u_2$, the element displacement field corresponds to rigid body motion and no straining of the element occurs. The first

derivative of Equation 6.3 with respect to x yields a constant value that, as we already know, represents the element axial strain. Hence, the truss element satisfies the completeness requirement, since both displacement and strain can take on constant values regardless of element size. Also note that the truss element satisfies the compatibility requirement automatically, since only displacement is involved, and displacement compatibility is enforced at the nodal connections via the system assembly procedure.

In light of the completeness requirement, we can now see that choice of the linear polynomial representation of the displacement field, Equation 6.2, was not arbitrary. Inclusion of the constant term a_0 ensures the possibility of rigid body motion, while the first-order term provides for a constant first derivative. Further, *only* two terms can be included in the representation, as only two boundary conditions have to be satisfied, corresponding to the two element degrees of freedom. Conversely, if the linear term were to be replaced by a quadratic term a_2x^2, for example, the coefficients could still be obtained to mathematically satisfy the nodal displacement conditions, but constant first derivative (other than a value of zero) could not be obtained under any circumstances.

Determination of the interpolation functions for the truss element, as just described, is quite simple. Nevertheless, the procedure is typical of that used to determine the interpolation functions for any element in which polynomials are utilized. Prior to examination of more complex elements, we revisit the development of the beam element interpolation functions with specific reference to the compatibility and completeness requirements. Recalling from Chapter 5 that the integral formulation (via Galerkin's method, Equation 5.49 for the two-dimensional beam element includes the second derivative of displacement, the compatibility condition requires that both displacement and the first derivative of displacement (slope) be continuous at the element boundaries. By including the slopes at element nodes as nodal variables in addition to nodal displacements, the compatibility condition is satisfied via the system assembly procedure. As we have seen, the beam element then has 4 degrees of freedom and the displacement field is represented as the cubic polynomial

$$v(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \tag{6.5}$$

which is ultimately to be expressed in terms of interpolation functions and nodal variables as

$$v(x) = N_1v_1 + N_2\theta_1 + N_3v_2 + N_4\theta_2 = [N_1 \quad N_2 \quad N_3 \quad N_4]\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{6.6}$$

Rewriting Equation 6.5 as the matrix product,

$$v(x) = [1 \quad x \quad x^2 \quad x^3]\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{6.7}$$

the nodal conditions

$$\begin{aligned} v(x=0) &= v_1 \\ \left.\frac{\mathrm{d}v}{\mathrm{d}x}\right|_{x=0} &= \theta_1 \\ v(x=L) &= v_2 \\ \left.\frac{\mathrm{d}v}{\mathrm{d}x}\right|_{x=L} &= \theta_2 \end{aligned} \tag{6.8}$$

are applied to obtain

$$v_1 = [1 \quad 0 \quad 0 \quad 0] \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{6.9}$$

$$\theta_1 = [0 \quad 1 \quad 0 \quad 0] \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{6.10}$$

$$v_2 = [1 \quad L \quad L^2 \quad L^3] \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{6.11}$$

$$\theta_2 = [0 \quad 1 \quad 2L \quad 3L^2] \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{6.12}$$

The last four equations are combined into the equivalent matrix form

$$\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{6.13}$$

The system represented by Equation 6.13 can be solved for the polynomial coefficients by inverting the coefficient matrix to obtain

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\dfrac{3}{L^2} & -\dfrac{2}{L} & \dfrac{3}{L^2} & -\dfrac{1}{L} \\ \dfrac{2}{L^3} & \dfrac{1}{L^2} & -\dfrac{2}{L^3} & \dfrac{1}{L^2} \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{6.14}$$

The interpolation functions can now be obtained by substituting the coefficients given by Equation 6.14 into Equation 6.5 and collecting coefficients of the nodal variables. However, the following approach is more direct and algebraically simpler. Substitute Equation 6.14 into Equation 6.7 and equate to Equation 6.6 to obtain

$$v(x) = [1 \quad x \quad x^2 \quad x^3] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\dfrac{3}{L^2} & -\dfrac{2}{L} & \dfrac{3}{L^2} & -\dfrac{1}{L} \\ \dfrac{2}{L^3} & \dfrac{1}{L^2} & -\dfrac{2}{L^3} & \dfrac{1}{L^2} \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$= [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{6.15}$$

The interpolation functions are

$$[N_1 \quad N_2 \quad N_3 \quad N_4] = [\,1 \quad x \quad x^2 \quad x^3\,] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\dfrac{3}{L^2} & -\dfrac{2}{L} & \dfrac{3}{L^2} & -\dfrac{1}{L} \\ \dfrac{2}{L^3} & \dfrac{1}{L^2} & -\dfrac{2}{L^3} & \dfrac{1}{L^2} \end{bmatrix} \tag{6.16}$$

and note that the results of Equation 6.16 are identical to those shown in Equation 4.26.

The reader may wonder why we repeat the development of the beam element interpolation functions. The purpose is twofold: (1) to establish a general procedure for use with polynomial representations of the field variable and (2) to revisit the beam element formulation in terms of compatibility and completeness requirements. The general procedure begins with expressing the field variable as a polynomial of order one fewer than the number of degrees of freedom exhibited by the element. Using the examples of the truss and beam elements, it has been shown that a two-node element may have 2 degrees of freedom, as in the truss element where only displacement continuity is required, or 4 degrees of freedom, as in the beam element where slope continuity is required. Next the nodal (boundary) conditions are applied and the coefficients of the polynomial are computed accordingly. Finally, the polynomial coefficients are substituted into the field variable representation in terms of nodal variables to obtain the explicit form of the interpolation functions.

Examination of the completeness condition for the beam element requires a more-detailed thought process. The polynomial representation of the displacement field is such that only the third derivative is *guaranteed* to have a constant

value, since any lower-order derivative involves the spatial variable. However, if we examine the conditions under which the element undergoes rigid body translation, for example, we find that the nodal forces must be of equal magnitude and the same sense and the applied nodal moments must be zero. Also, for rigid body translation, the slopes at the nodes of the element are zero. In such case, the second derivative of deflection, directly proportional to bending moment, is zero and the shear force, directly related to the third derivative of deflection, is constant. (Simply recall the shear force and bending moment relations from the mechanics of materials theory.) Therefore, the field variable representation as a cubic polynomial allows rigid body translation. In the case of the beam element, we must also verify the possibility of rigid body rotation. This consideration, as well as those of constant bending moment and shear force, is left for end-of-chapter problems.

6.3.1 Higher-Order One-Dimensional Elements

In formulating the truss element and the one-dimensional heat conduction element (Chapter 5), only line elements having a single degree of freedom at each of two nodes are considered. While quite appropriate for the problems considered, the linear element is by no means the only one-dimensional element that can be formulated for a given problem type. Figure 6.2 depicts a three-node line element in which node 2 is an *interior* node. As mentioned briefly in Chapter 1, an interior node is *not* connected to any other node in any other element in the model. Inclusion of the interior node is a mathematical tool to increase the order of approximation of the field variable. Assuming that we deal with only 1 degree of freedom at each node, the appropriate polynomial representation of the field variable is

$$\phi(x) = a_0 + a_1 x + a_2 x^2 \tag{6.17}$$

and the nodal conditions are

$$\begin{aligned} \phi(x = 0) &= \phi_1 \\ \phi\left(x = \frac{L}{2}\right) &= \phi_2 \\ \phi(x = L) &= \phi_3 \end{aligned} \tag{6.18}$$

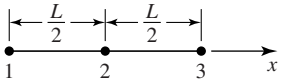

Figure 6.2 A three-node line element. Node 2 is an interior node.

Applying the general procedure outlined previously in the context of the beam element, we apply the nodal (boundary) conditions to obtain

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{L}{2} & \frac{L^2}{4} \\ 1 & L & L^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} \tag{6.19}$$

from which the interpolation functions are obtained via the following sequence

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{L} & \frac{4}{L} & -\frac{1}{L} \\ \frac{2}{L^2} & -\frac{4}{L^2} & \frac{2}{L^2} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} \tag{6.20a}$$

$$\begin{aligned} \phi(x) &= [1 \quad x \quad x^2] \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{L} & \frac{4}{L} & -\frac{1}{L} \\ \frac{2}{L^2} & -\frac{4}{L^2} & \frac{2}{L^2} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} \\ &= [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} \end{aligned} \tag{6.20b}$$

$$\begin{aligned} N_1(x) &= 1 - \frac{3}{L}x + \frac{2}{L^2}x^2 \\ N_2(x) &= \frac{4x}{L}\left(1 - \frac{x}{L}\right) \\ N_3(x) &= \frac{x}{L}\left(\frac{2x}{L} - 1\right) \end{aligned} \tag{6.20c}$$

Note that each interpolation function varies quadratically in x and has value of unity at its associated node and value zero at the other two nodes, as illustrated in Figure 6.3. These observations lead to a shortcut method of concocting the interpolation functions for a C^0 line element as products of monomials as follows. Let $s = x/L$ such that $s_1 = 0, s_2 = 1/2, s_3 = 1$ are the nondimensional coordinates of nodes 1, 2, and 3, respectively. Instead of following the formal procedure used previously, we hypothesize, for example,

$$N_1(s) = C_1(s - s_2)(s - s_3) \tag{6.21}$$

where C_1 is a constant. The first monomial term ensures that N_1 has a value of zero at node 2 and the second monomial term ensures the same at node 3. Therefore, we need to determine only the value of C_1 to provide unity value at node 1.

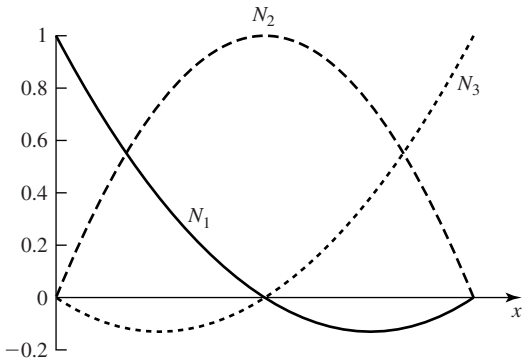

Figure 6.3 Spatial variation of interpolation functions for a three-node line element.

Substituting $s = 0$, we obtain

$$N_1(s = 0) = 1 = C_1\left(0 - \frac{1}{2}\right)(0 - 1) \tag{6.22}$$

yielding $C_1 = 2$ and

$$N_1(s) = 2\left(s - \frac{1}{2}\right)(s - 1) \tag{6.23}$$

Following similar logic and procedure shows that

$$N_2(s) = -4s(s - 1) \tag{6.24}$$

$$N_3(s) = 2s\left(s - \frac{1}{2}\right) \tag{6.25}$$

Substituting $s = x/L$ in Equations 6.23–6.25 and expanding shows that the results are identical to those given in Equation 6.20. The monomial-based procedure can be extended to line elements of any order as illustrated by the following example.

EXAMPLE 6.1

Use the monomial method to obtain the interpolation functions for the four-node line element shown in Figure 6.4.

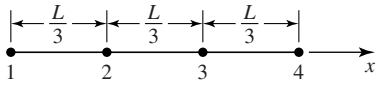

Figure 6.4 Four-node line element of Example 6.1.

■ **Solution**

Using $s = x/L$, we have $s_1 = 0$, $s_2 = 1/3$, $s_3 = 2/3$, and $s_4 = 1$. The monomial terms of interest are s, $s - 1/3$, $s - 2/3$, and $s - 1$. The monomial products

$$N_1(s) = C_1\left(s - \frac{1}{3}\right)\left(s - \frac{2}{3}\right)(s - 1)$$

$$N_2(s) = C_2 s\left(s - \frac{2}{3}\right)(s - 1)$$

$$N_3(s) = C_3 s\left(s - \frac{1}{3}\right)(s - 1)$$

$$N_4(s) = C_4 s\left(s - \frac{1}{3}\right)\left(s - \frac{2}{3}\right)$$

automatically satisfy the required zero-value conditions for each interpolation function. Hence, we need evaluate only the constants C_i such that $N_i(s = s_i) = 1$, $i = 1, 4$. Applying each of the four unity-value conditions, we obtain

$$N_1(0) = 1 = C_1\left(-\frac{1}{3}\right)\left(-\frac{2}{3}\right)(-1)$$

$$N_2\left(\frac{1}{3}\right) = 1 = C_2\left(\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{2}{3}\right)$$

$$N_3\left(\frac{2}{3}\right) = 1 = C_3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(-\frac{1}{3}\right)$$

$$N_4(1) = 1 = C_4(1)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)$$

from which $C_1 = -\frac{9}{2}, C_2 = \frac{27}{2}, C_3 = -\frac{27}{2}, C_4 = \frac{9}{2}$.

The interpolation functions are then given as

$$N_1(s) = -\frac{9}{2}\left(s - \frac{1}{3}\right)\left(s - \frac{2}{3}\right)(s - 1)$$

$$N_2(s) = \frac{27}{2} s\left(s - \frac{2}{3}\right)(s - 1)$$

$$N_3(s) = -\frac{27}{2} s\left(s - \frac{1}{3}\right)(s - 1)$$

$$N_4(s) = \frac{9}{2} s\left(s - \frac{1}{3}\right)\left(s - \frac{2}{3}\right)$$

6.4 POLYNOMIAL FORMS: GEOMETRIC ISOTROPY

The previous discussion of one-dimensional (line) elements revealed that the polynomial representation of the field variable must contain the same number of terms as the number of nodal degrees of freedom. In addition, to satisfy the completeness requirement, the polynomial representation for an M-degree of freedom element should contain all powers of the independent variable up to and including $M - 1$. Another way of stating the latter requirement is that the polynomial is *complete*. In two and three dimensions, polynomial representations of the field variable, in general, satisfy the compatibility and completeness requirements if the polynomial exhibits the property known as *geometric isotropy* [1]. A mathematical function satisfies geometric isotropy if the functional form does not change under a translation or rotation of coordinates. In two dimensions, a complete polynomial of order M can be expressed as

$$P_M(x, y) = \sum_{k=0}^{N_t^{(2)}} a_k x^i y^j \qquad i + j \leq M \tag{6.26}$$

where $N_t^{(2)} = [(M + 1)(M + 2)]/2$ is the total number of terms. A complete polynomial as expressed by Equation 6.26 satisfies the condition of geometric isotropy, since the two variables, x and y, are included in each term in similar powers. Therefore, a translation or rotation of coordinates is not prejudicial to either independent variable.

A graphical method of depicting complete two-dimensional polynomials is the so-called Pascal triangle shown in Figure 6.5. Each horizontal line represents a polynomial of order M. A complete polynomial of order M must contain all terms shown above the horizontal line. For example, a complete quadratic polynomial in two dimensions must contain six terms. Hence, for use in a finite element representation of a field variable, a complete quadratic expression requires six nodal degrees of freedom in the element. We examine this particular case in the context of triangular elements in the next section.

In addition to the complete polynomials, incomplete polynomials also exhibit geometric isotropy if the incomplete polynomial is *symmetric*. In this context, symmetry implies that the independent variables appear as "equal and

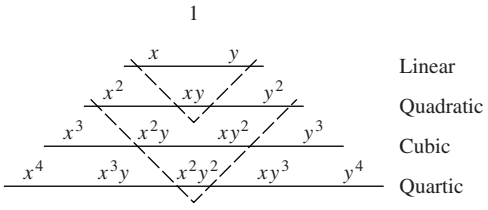

Figure 6.5 Pascal triangle for polynomials in two dimensions.

opposite pairs," ensuring that each independent variable plays an equal role in the polynomial. For example, the four-term incomplete quadratic polynomial

$$P(x, y) = a_0 + a_1 x + a_2 y + a_3 x^2 \tag{6.27}$$

is not symmetric, as there is a quadratic term in x but the corresponding quadratic term in y does not appear. On the other hand, the incomplete quadratic polynomial

$$P(x, y) = a_0 + a_1 x + a_2 y + a_3 xy \tag{6.28}$$

is symmetric, as the quadratic term gives equal "weight" to both variables.

A very convenient way of visualizing some of the commonly used incomplete but symmetric polynomials of a given order is also afforded by the Pascal triangle. Again referring to Figure 6.5, the dashed lines show the terms that must be included in an incomplete yet symmetric polynomial of a given order. (These are, of course, not the only incomplete, symmetric polynomials that can be constructed.) All terms above the dashed lines must be included in a polynomial representation if the function is to exhibit geometric isotropy. This feature of polynomials is utilized to a significant extent in following the development of various element interpolation functions.

As in the two-dimensional case, to satisfy the geometric isotropy requirements, the polynomial expression of the field variable in three dimensions must be complete or incomplete but symmetric. Completeness and symmetry can also be depicted graphically by the "Pascal pyramid" shown in Figure 6.6. While the three-dimensional case is a bit more difficult to visualize, the basic premise

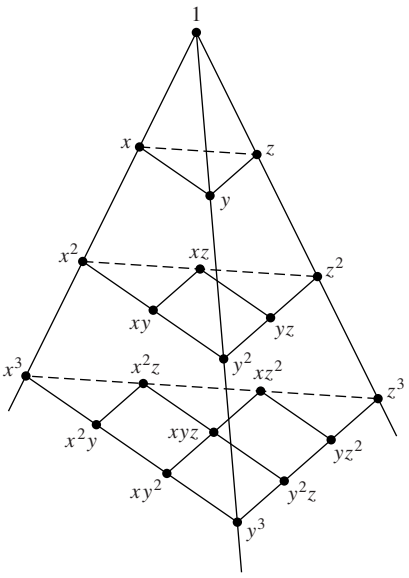

Figure 6.6 Pascal "pyramid" for polynomials in three dimensions.

remains that each independent variable must be of equal "strength" in the polynomial. For example, the 3-D quadratic polynomial

$$P(x, y, z) = a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 + a_6z^2 + a_7xy + a_8xz + a_9yz \tag{6.29}$$

is complete and could be applied to an element having 10 nodes. Similarly, an incomplete, symmetric form such as

$$P(x, y, z) = a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 + a_6z^2 \tag{6.30}$$

or

$$P(x, y, z) = a_0 + a_1x + a_2y + a_3z + a_4xy + a_5xz + a_6yz \tag{6.31}$$

could be used for elements having seven nodal degrees of freedom (an unlikely case, however).

Geometric isotropy is not an *absolute* requirement for field variable repesentation [1], hence, interpolation functions. As demonstrated by many researchers, incomplete representations are quite often used and solution convergence attained. However, in terms of *h*-refinement, use of geometrically isotropic representations guarantees satisfaction of the compatibility and completeness requirements. For the *p*-refinement method, the reader is reminded that the interpolation functions in any finite element analysis solution are approximations to the power series expansion of the problem solution. As we increase the number of element nodes, the order of the interpolation functions increases and, in the limit, as the number of nodes approaches infinity, the polynomial expression of the field variable approaches the power series expansion of the solution.

6.5 TRIANGULAR ELEMENTS

The interpolation functions for triangular elements are inherently formulated in two dimensions and a family of such elements exists. Figure 6.7 depicts the first three elements (linear, quadratic, and cubic) of the family. Note that, in the case of the cubic element, an internal node exists. The internal node is required to

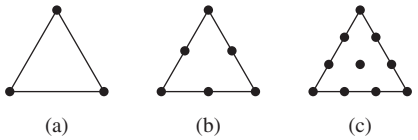

Figure 6.7 Triangular elements: (a) 3-node linear, (b) 6-node quadratic, (c) 10-node cubic.

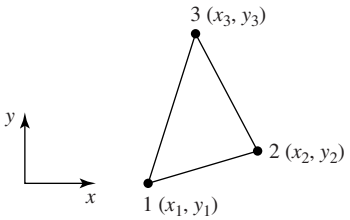

Figure 6.8 A general three-node triangular element referred to global coordinates.

obtain geometric isotropy, as is subsequently discussed. The triangular elements are not limited to two-dimensional problems. In fact, the triangular elements can be used in axisymmetric 3-D cases (discussed later in this chapter) as well as in structural analyses involving out-of-plane bending, as in plate and shell structures. In the latter cases, the nodal degrees of freedom include first derivatives of the field variable as well as the field variable itself. While plate and shell problems are beyond the scope of this book, we allude to those problems again briefly in Chapter 9.

Figure 6.8 depicts a general, three-node triangular element to which we attach an element coordinate system that is, for now, assumed to be the same as the global system. Here, it is assumed that only 1 degree of freedom is associated with each node. We express the field variable in the polynomial form

$$\phi(x, y) = a_0 + a_1 x + a_2 y \tag{6.32}$$

Applying the nodal conditions

$$\begin{aligned} \phi(x_1, y_1) &= \phi_1 \\ \phi(x_2, y_2) &= \phi_2 \\ \phi(x_3, y_3) &= \phi_3 \end{aligned} \tag{6.33}$$

and following the general procedure previously outlined, we obtain

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_2 & y_2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} \tag{6.34}$$

To solve for the polynomial coefficients, the matrix of coefficients in Equation 6.34 must be inverted. Inversion of the matrix is algebraically tedious but straightforward, and we find

$$a_0 = \frac{1}{2A}[\phi_1(x_2 y_3 - x_3 y_2) + \phi_2(x_3 y_1 - x_1 y_3) + \phi_3(x_1 y_2 - x_2 y_1)]$$

$$a_1 = \frac{1}{2A}[\phi_1(y_2 - y_3) + \phi_2(y_3 - y_1) + \phi_3(y_1 - y_2)]$$
$$a_2 = \frac{1}{2A}[\phi_1(x_3 - x_2) + \phi_2(x_1 - x_3) + \phi_3(x_2 - x_1)] \tag{6.35}$$

Substituting the values into Equation 6.32 and collecting coefficients of the nodal variables, we obtain

$$\phi(x, y) = \frac{1}{2A}\left\{\begin{array}{l}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]\phi_1 \\ \quad + [(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y]\phi_2 \\ \quad + [(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y]\phi_3\end{array}\right\} \tag{6.36}$$

Given the form of Equation 6.36, the interpolation functions are observed to be

$$N_1(x, y) = \frac{1}{2A}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$
$$N_2(x, y) = \frac{1}{2A}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y] \tag{6.37}$$
$$N_3(x, y) = \frac{1}{2A}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y]$$

where A is the area of the triangular element. Given the coordinates of the three vertices of a triangle, it can be shown that the area is given by

$$A = \frac{1}{2}\begin{vmatrix}1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3\end{vmatrix} \tag{6.38}$$

Note that the algebraically complex form of the interpolation functions arises primarily from the choice of the element coordinate system of Figure 6.8. As the linear representation of the field variable exhibits geometric isotropy, location and orientation of the element coordinate axes can be chosen arbitrarily without affecting the interpolation results. If, for example, the element coordinate system shown in Figure 6.9 is utilized, considerable algebraic simplification results. In the coordinate system shown, we have $x_1 = y_1 = y_2 = 0$, $2A = x_2y_3$,

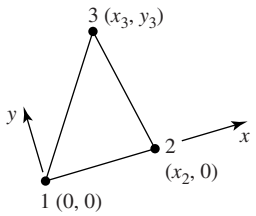

Figure 6.9 Three-node triangle having an element coordinate system attached to the element.

and the interpolation functions become

$$N_1(x, y) = \frac{1}{x_2 y_3}[x_2 y_3 - y_3 x + (x_3 - x_2) y]$$

$$N_2(x, y) = \frac{1}{x_2 y_3}[y_3 x - x_3 y] \tag{6.39}$$

$$N_3(x, y) = \frac{y}{y_3}$$

which are clearly of a simpler form than Equation 6.37. The simplification is not without cost, however. If the element coordinate system is directly associated with element orientation, as in Figure 6.9, the element characteristic matrices must be transformed to a common global coordinate system during model assembly. (Recall the transformation of stiffness matrices demonstrated for bar and beam elements earlier.) As finite element models usually employ a large number of elements, the additional computations required for element transformation can be quite time consuming. Consequently, computational efficiency is improved if each element coordinate system is oriented such that the axes are parallel to the global axes. The transformation step is then unnecessary when model assembly takes place. In practice, most commercial finite element software packages provide for use of either type element coordinate as a user option [6].

Returning to Equation 6.32, observe that it is possible for the field variable to take on a constant value, as per the completeness requirement, and that the first partial derivatives with respect to the independent variables x and y are constants. The latter shows that the gradients of the field variable are constant in both coordinate directions. For a planar structural element, this results in constant strain components. In fact, in structural applications, the three-node triangular element is commonly known as a *constant strain triangle* (CST, for short). In the case of heat transfer, the element produces constant temperature gradients, therefore, constant heat flow within an element.

6.5.1 Area Coordinates

When expressed in Cartesian coordinates, the interpolation functions for the triangular element are algebraically complex. Further, the integrations required to obtain element characteristic matrices are cumbersome. Considerable simplification of the interpolation functions as well as the subsequently required integration is obtained via the use of *area coordinates*. Figure 6.10 shows a three-node triangular element divided into three areas defined by the nodes and an arbitrary interior point $P(x, y)$. Note: P is *not* a node. The area coordinates of P are defined as

$$L_1 = \frac{A_1}{A} \qquad L_2 = \frac{A_2}{A} \qquad L_3 = \frac{A_3}{A} \tag{6.40}$$

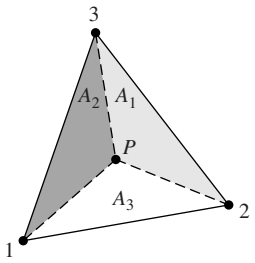

Figure 6.10 Areas used to define area coordinates for a triangular element.

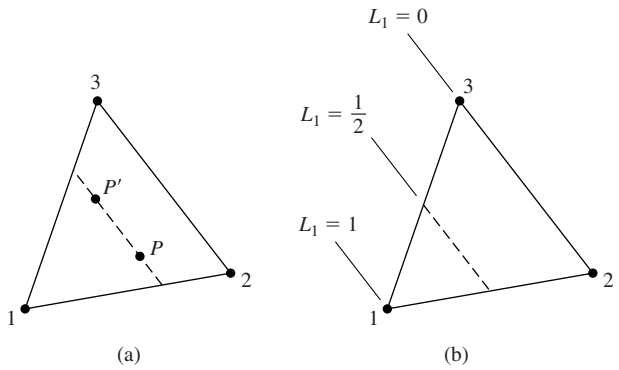

Figure 6.11
(a) Area A_1 associated with either P or P' is constant.
(b) Lines of the constant area coordinate L_1.

where A is the total area of the triangle. Clearly, the area coordinates are not independent, since

$$L_1 + L_2 + L_3 = 1 \tag{6.41}$$

The dependency is to be expected, since Equation 6.40 expresses the location of a point in two-dimensions using three coordinates.

The important properties of area coordinates for application to triangular finite elements are now examined with reference to Figure 6.11. In Figure 6.11a, a dashed line parallel to the side defined by nodes 2 and 3 is indicated. For any two points P and P' on this line, the areas of the triangles formed by nodes 2 and 3 and either P or P' are identical. This is because the base and height of any triangle so formed are constants. Further, as the dashed line is moved closer to node 1, area A_1 increases linearly and has value $A_1 = A$, when evaluated at node 1. Therefore, area coordinate L_1 is constant on any line parallel to the side of the triangle opposite node 1 and varies linearly from a value of unity at node 1 to value of zero along the side defined by nodes 2 and 3, as depicted in Figure 6.11b. Similar arguments can be made for the behavior of L_2 and L_3. These observations

can be used to write the following conditions satisfied by the area coordinates when evaluated at the nodes:

$$\begin{aligned} &\text{Node 1:} \quad L_1 = 1 \qquad L_2 = L_3 = 0 \\ &\text{Node 2:} \quad L_2 = 1 \qquad L_1 = L_3 = 0 \\ &\text{Node 3:} \quad L_3 = 1 \qquad L_1 = L_2 = 0 \end{aligned} \tag{6.42}$$

The conditions expressed by Equation 6.42 are exactly the conditions that must be satisfied by interpolation functions at the nodes of the triangular element. So, we express the field variable as

$$\phi(x, y) = L_1\phi_1 + L_2\phi_2 + L_3\phi_3 \tag{6.43}$$

in terms of area coordinates. Is this different from the field variable representation of Equation 6.36? If the area coordinates are expressed explicitly in terms of the nodal coordinates, the two field variable representations are shown to be identical. The true advantages of area coordinates are seen more readily in developing interpolation functions for higher-order elements and performing integration of various forms of the interpolation functions.

6.5.2 Six-Node Triangular Element

A six-node element is shown in Figure 6.12a. The additional nodes 4, 5, and 6 are located at the midpoints of the sides of the element. As we have six nodes, a complete polynomial representation of the field variable is

$$\phi(x, y) = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 \tag{6.44}$$

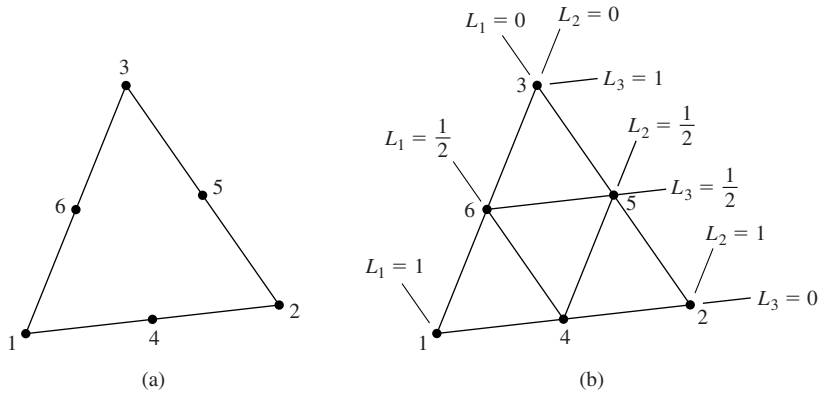

Figure 6.12 Six-node triangular elements: (a) Node numbering convention. (b) Lines of constant values of the area coordinates.

which is ultimately to be expressed in terms of interpolation functions and nodal values as

$$\phi(x, y) = \sum_{i=1}^{6} N_i(x, y)\phi_i \tag{6.45}$$

As usual, each interpolation function must be such that its value is unity when evaluated at its associated node and zero when evaluated at any of the other five nodes. Further, each interpolation is a quadratic function, since the field variable representation is quadratic.

Figure 6.12b shows the six-node element with lines of constant values of the area coordinates passing through the nodes. Using this figure and a bit of logic, the interpolation functions are easily "constructed" in area coordinates. For example, interpolation function $N_1(x, y) = N_1(L_1, L_2, L_3)$ must have value of zero at nodes 2, 3, 4, 5, and 6. Noting that $L_1 = 1/2$ at nodes 4 and 6, inclusion of the term $L_1 - 1/2$ ensures a zero value at those two nodes. Similarly, $L_1 = 0$ at nodes 2, 3, 4, so the term L_1 satisfies the conditions at those three nodes. Therefore, we propose

$$N_1 = L_1\left(L_1 - \frac{1}{2}\right) \tag{6.46}$$

However, evaluation of Equation 6.46 at node 1, where $L_1 = 1$, results in $N_1 = 1/2$. As N_1 must be unity at node 1, Equation 6.46 is modified to

$$N_1 = 2L_1\left(L_1 - \frac{1}{2}\right) = L_1(2L_1 - 1) \tag{6.47a}$$

which satisfies the required conditions at each of the six nodes and is a quadratic function, since L_1 is a linear function of x and y.

Applying the required nodal conditions to the remaining five interpolation functions in turn, we obtain

$$N_2 = L_2(2L_2 - 1) \tag{6.47b}$$

$$N_3 = L_3(2L_3 - 1) \tag{6.47c}$$

$$N_4 = 4L_1L_2 \tag{6.47d}$$

$$N_5 = 4L_2L_3 \tag{6.47e}$$

$$N_6 = 4L_1L_3 \tag{6.47f}$$

Using a similarly straightforward procedure, interpolation functions for additional higher-order triangular elements can be constructed. The 10-node cubic element is left as an exercise.

6.5.3 Integration in Area Coordinates

As seen in Chapter 5 and encountered again in later chapters, integration of various forms of the interpolation functions over the domain of an element are

required in formulating element characteristic matrices and load vectors. When expressed in area coordinates, integrals of the form

$$\iint_A L_1^a L_2^b L_3^c \, dA \tag{6.48}$$

(where A is the total area of a triangle defined by nodes 1, 2, 3) must often be evaluated. The relation

$$\iint_A L_1^a L_2^b L_3^c \, dA = (2A)\frac{a!b!c!}{(a+b+c+2)!} \tag{6.49}$$

has been shown [7] to be valid for all exponents a, b, c that are positive integers (recall $0! = 1$). Therefore, integration in area coordinates is quite straightforward.

EXAMPLE 6.2

As will be shown in Chapter 7, the convection terms of the stiffness matrix for a 2-D heat transfer element are of the form

$$k_{ij} = \int_A h N_i N_j \, dA$$

where h is the convection coefficient and A is element area. Use the interpolation functions for a six-node triangular element given by Equation 6.47 to compute k_{24}.

■ **Solution**

Using Equation 6.47b and 6.47d, we have

$$N_2 = L_2(2L_2 - 1)$$
$$N_4 = 4L_1L_2$$

so (assuming h is a constant)

$$k_{24} = h\int_A L_2(2L_2 - 1)4L_1L_2 \, dA = h\int_A \left(8L_1L_2^3 - 4L_1L_2^2\right) dA$$

Applying Equation 6.49, we have

$$h\int_A 8L_1L_2^3 \, dA = 8h(2A)\frac{(1!)(3!)(0!)}{(1+3+0+2)!} = \frac{96hA}{720} = \frac{2hA}{15}$$

$$h\int_A 4L_1L_2^2 \, dA = 4h(2A)\frac{(1!)(2!)(0!)}{(1+2+0+2)!} = \frac{16hA}{120} = \frac{2hA}{15}$$

Therefore,

$$k_{24} = \frac{2hA}{15} - \frac{2hA}{15} = 0$$

6.6 RECTANGULAR ELEMENTS

Rectangular elements are convenient for use in modeling regular geometries, can be used in conjunction with triangular elements, and form the basis for development of general quadrilateral elements. The simplest of the rectangular family of elements is the four-node rectangle shown in Figure 6.13, where it is assumed that the sides of the rectangular are parallel to the global Cartesian axes. By convention, we number the nodes sequentially in a counterclockwise direction, as shown. As there are four nodes and 4 degrees of freedom, a four-term polynomial expression for the field variable is appropriate. Since there is no complete four-term polynomial in two dimensions, the incomplete, symmetric expression

$$\phi(x, y) = a_0 + a_1 x + a_2 y + a_3 xy \tag{6.50}$$

is used to ensure geometric isotropy. Applying the four nodal conditions and writing in matrix form gives

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{6.51}$$

which formally gives the polynomial coefficients as

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{bmatrix}^{-1} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \tag{6.52}$$

In terms of the nodal values, the field variable is then described by

$$\phi(x, y) = [1 \quad x \quad y \quad xy]\{a\} = [1 \quad x \quad y \quad xy] \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{bmatrix}^{-1} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \tag{6.53}$$

from which the interpolation functions can be deduced.

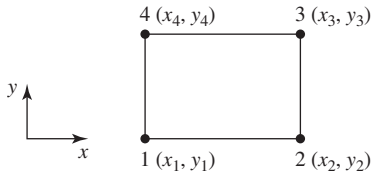

Figure 6.13 A four-node rectangular element defined in global coordinates.

The form of Equation 6.53 suggests that expression of the interpolation functions in terms of the nodal coordinates is algebraically complex. Fortunately, the complexity can be reduced by a more judicious choice of coordinates. For the rectangular element, we introduce the normalized coordinates (also known as *natural coordinates* or *serendipity coordinates*) r and s as

$$r = \frac{x - \bar{x}}{a} \qquad s = \frac{y - \bar{y}}{b} \tag{6.54}$$

where $2a$ and $2b$ are the width and height of the rectangle, respectively, and the coordinates of the centroid are

$$\bar{x} = \frac{x_1 + x_2}{2} \qquad \bar{y} = \frac{y_1 + y_4}{2} \tag{6.55}$$

as shown in Figure 6.14a. Therefore, r and s are such that the values range from -1 to $+1$, and the nodal coordinates are as in Figure 6.14b.

Applying the conditions that must be satisfied by each interpolation function at each node, we obtain (essentially by inspection)

$$\begin{aligned} N_1(r, s) &= \frac{1}{4}(1 - r)(1 - s) \\ N_2(r, s) &= \frac{1}{4}(1 + r)(1 - s) \\ N_3(r, s) &= \frac{1}{4}(1 + r)(1 + s) \\ N_4(r, s) &= \frac{1}{4}(1 - r)(1 + s) \end{aligned} \tag{6.56a}$$

hence

$$\phi(x, y) = \phi(r, s) = N_1(r, s)\phi_1 + N_2(r, s)\phi_2 + N_3(r, s)\phi_3 + N_4(r, s)\phi_4 \tag{6.56b}$$

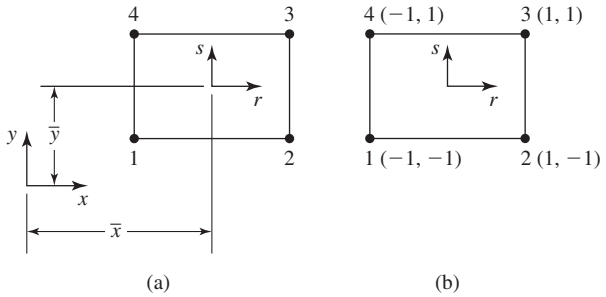

Figure 6.14 A four-node rectangular element showing (a) the translation to natural coordinates, (b) the natural coordinates of each node.

As in the case of triangular elements using area coordinates, the interpolation functions are much simpler algebraically when expressed in terms of the natural coordinates. Nevertheless, all required conditions are satisfied and the functional form is identical to that used to express the field variable in Equation 6.50. Also as with area coordinates, integrations involving the interpolation functions expressed in the natural coordinates are simplified, since the integrands are relatively simple polynomials (for rectangular elements) and the integration limits (when integrating over the area of the element) are –1 and +1. Further discussion of such integration requirements, particularly numerical integration techniques, is postponed until later in this chapter.

To develop a higher-order rectangular element, the logical progression is to place an additional node at the midpoint of each side of the element, as in Figure 6.15. This poses an immediate problem, however. Inspection of the Pascal triangle shows that we cannot construct a complete polynomial having eight terms, but we have a choice of two incomplete, symmetric cubic polynomials:

$$\phi(x, y) = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + a_6x^3 + a_7y^3 \tag{6.57a}$$

$$\phi(x, y) = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + a_6x^2y + a_7xy^2 \tag{6.57b}$$

Rather than grapple with choosing one or the other, we use the natural coordinates and the nodal conditions that must be satisfied by each interpolation function to obtain the functions serendipitously. For example, interpolation function N_1 must evaluate to zero at all nodes except node 1, where its value must be unity. At nodes 2, 3, and 6, $r = 1$, so including the term $r - 1$ satisfies the zero condition at those nodes. Similarly, at nodes 4 and 7, $s = 1$ so the term $s - 1$ ensures the zero condition at those two nodes. Finally, at node 5, $(r, s) = (0, -1)$, and at node 8, $(r, s) = (-1, 0)$. Hence, at nodes 5 and 8, the term $r + s + 1$ is identically zero. Using this reasoning, the interpolation function associated with node 1 is to be of the form

$$N_1(r, s) = (1 - r)(1 - s)(r + s + 1) \tag{6.58}$$

Evaluating at node 1 where $(r, s) = (-1, -1)$, we obtain $N_1 = -4$, so a correction is required to obtain the unity value. The final form is then

$$N_1(r, s) = \frac{1}{4}(r - 1)(1 - s)(r + s + 1) \tag{6.59a}$$

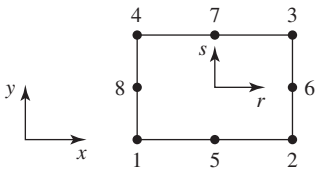

Figure 6.15 Eight-node rectangular element showing both global and natural coordinate axes.

A parallel procedure for the interpolation functions associated with the other three corner nodes leads to

$$N_2(r, s) = \frac{1}{4}(r + 1)(1 - s)(s - r + 1) \tag{6.59b}$$

$$N_3(r, s) = \frac{1}{4}(1 + r)(1 + s)(r + s - 1) \tag{6.59c}$$

$$N_4(r, s) = \frac{1}{4}(r - 1)(1 + s)(r - s + 1) \tag{6.59d}$$

The form of the interpolation functions associated with the midside nodes is simpler to obtain than those for the corner nodes. For example, N_5 has a value of zero at nodes 2, 3, and 6 if it contains the term $r - 1$ and is also zero at nodes 1, 4, and 8 if the term $1 + r$ is included. Finally, if a zero value is at node 7, $(r, s) = (0, 1)$ is obtained by inclusion of $s - 1$. The form for N_5 is

$$N_5 = \frac{1}{2}(1 - r)(1 + r)(1 - s) = \frac{1}{2}(1 - r^2)(1 - s) \tag{6.59e}$$

where the leading coefficient ensures a unity value at node 5. For the other midside nodes,

$$N_6 = \frac{1}{2}(1 + r)(1 - s^2) \tag{6.59f}$$

$$N_7 = \frac{1}{2}(1 - r^2)(1 + s) \tag{6.59g}$$

$$N_8 = \frac{1}{2}(1 - r)(1 - s^2) \tag{6.59h}$$

are determined in the same manner.

Many other, successively higher-order, rectangular elements have been developed [1]. In general, these higher-order elements include internal nodes that, in modeling, are troublesome, as they cannot be connected to nodes of other elements. The internal nodes are eliminated mathematically. The elimination process is such that the mechanical effects of the internal nodes are assigned appropriately to the external nodes.

6.7 THREE-DIMENSIONAL ELEMENTS

As in the two-dimensional case, there are two main families of three-dimensional elements. One is based on extension of triangular elements to tetrahedrons and the other on extension of rectangular elements to rectangular parallelopipeds

(often simply called *brick* elements). The algebraically cumbersome techniques for deriving interpolation functions in global Cartesian coordinates has been illustrated for two-dimensional elements. Those developments are not repeated here for three-dimensional elements; the procedures are algebraically identical but even more complex. Instead, we utilize only the more amenable approach of using natural coordinates to develop the interpolation functions for the two basic elements of the tetrahedral and brick families.

6.7.1 Four-Node Tetrahedral Element

A four-node tetrahedral element is depicted in Figure 6.16 in relation to a global Cartesian coordinate system. The nodes are numbered 1–4 per the convention that node 1 can be selected arbitrarily and nodes 2–4 are then specified in a counterclockwise direction from node 1. (This convention is the same as used by most commercial finite element analysis software and is very important in assuring geometrically correct tetrahedrons. On the other hand, tetrahedral element definition for finite element models is so complex that it is almost always accomplished by automeshing capabilities of specific software packages.)

In a manner analogous to use of area coordinates, we now introduce the concept of *volume coordinates* using Figure 6.17. Point $P(x, y, z)$ is an arbitrary point in the tetrahedron defined by the four nodes. As indicated by the dashed lines, point P and the four nodes define four other tetrahedra having volumes

$$\begin{aligned} V_1 &= \text{vol}(P234) \qquad V_2 = \text{vol}(P134) \\ V_3 &= \text{vol}(P124) \qquad V_4 = \text{vol}(P123) \end{aligned} \tag{6.60}$$

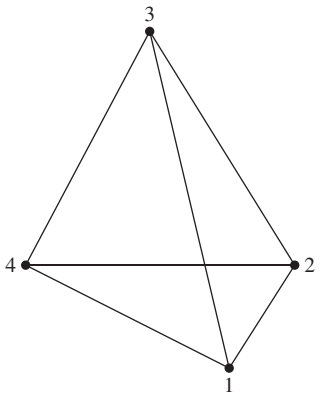

Figure 6.16 A four-node tetrahedral element.

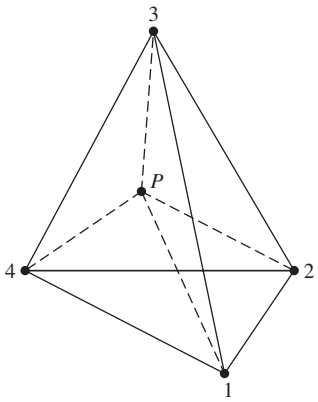

Figure 6.17 A four-node tetrahedral element, showing an arbitrary interest point defining four volumes.

The volume coordinates are defined as

$$\begin{aligned} L_1 &= \frac{V_1}{V} \\ L_2 &= \frac{V_2}{V} \\ L_3 &= \frac{V_3}{V} \\ L_4 &= \frac{V_4}{V} \end{aligned} \tag{6.61}$$

where V is the total volume of the element given by

$$V = \frac{1}{6}\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix} \tag{6.62}$$

As with area coordinates, the volume coordinates are not independent, since

$$V_1 + V_2 + V_3 + V_4 = V \tag{6.63}$$

Now let us examine the variation of the volume coordinates through the element. If, for example, point P corresponds to node 1, we find $V_1 = V$, $V_2 = V_3 = V_4 = 0$. Consequently $L_1 = 1$, $L_2 = L_3 = L_4 = 0$ at node 1. As P moves away from node 1, V_1 decreases linearly, since the volume of a tetrahedron is directly proportional to its height (the perpendicular distance from P to the plane defined by nodes 2, 3, and 4) and the area of its base (the triangle formed by nodes 2, 3, and 4). On any plane parallel to the base triangle of nodes 2, 3, 4, the value of L_1 is constant. Of particular importance is that, if P lies in the plane of nodes 2, 3, 4, the value of L_1 is zero. Identical observations apply to volume coordinates L_2, L_3, and L_4. So the volume coordinates satisfy all required nodal conditions for interpolation functions, and we can express the field variable as

$$\phi(x, y, z) = L_1\phi_1 + L_2\phi_2 + L_3\phi_3 + L_4\phi_4 \tag{6.64}$$

Explicit representation of the interpolation functions (i.e., the volume coordinates) in terms of global coordinates is, as stated, algebraically complex but straightforward. Fortunately, such explicit representation is not generally required, as element formulation can be accomplished using volume coordinates only. As with area coordinates, integration of functions of volume coordinates (required in developing element characteristic matrices and load vectors) is relatively simple. Integrals of the form

$$\iiint_V L_1^a L_2^b L_3^c L_4^d \, \mathrm{d}V \tag{6.65}$$

where a, b, c, d are positive integers and V is total element volume, appear in element formulation for various physical problems. As with area coordinates, integration in volume coordinates is straightforward [7], and we have the integration formula

$$\iiint_V L_1^a L_2^b L_3^c L_4^d \, dV = \frac{a!b!c!d!}{(a+b+c+d+3)!}(6V) \tag{6.66}$$

which is the three-dimensional analogy to Equation 6.49.

As another analogy with the two-dimensional triangular elements, the tetrahedral element is most useful in modeling irregular geometries. However, the tetrahedral element is not particularly amenable to use in conjunction with other element types, strictly as a result of the nodal configurations. This incompatibility is discussed in the following sections. As a final comment on the four-node tetrahedral element, we note that the field variable representation, as given by Equation 6.64, is a linear function of the Cartesian coordinates. Therefore, all the first partial derivatives of the field variable are constant. In structural applications, the tetrahedral element is a constant strain element; in general, the element exhibits constant gradients of the field variable in the coordinate directions.

Other elements of the tetrahedral family are depicted in Figure 6.18. The interpolation functions for the depicted elements are readily written in volume coordinates, as for higher-order two-dimensional triangular elements. Note particularly that the second element of the family has 10 nodes and a cubic form for the field variable and interpolation functions. A quadratic tetrahedral element cannot be constructed to exhibit geometric isotropy even if internal nodes are included.

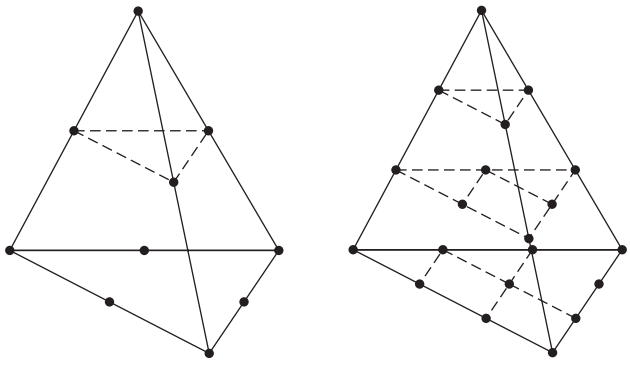

Figure 6.18 Higher-order tetrahedral elements: (a) 10 node. (b) 20 node.

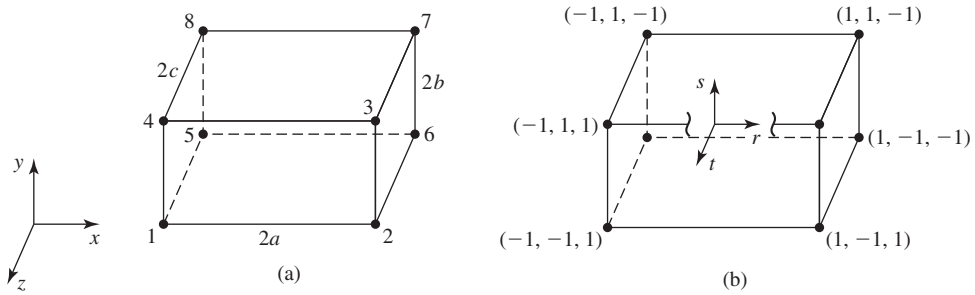

Figure 6.19 Eight-node brick element: (a) Global Cartesian coordinates. (b) Natural coordinates with an origin at the centroid.

6.7.2 Eight-Node Brick Element

The so-called eight node brick element (rectangular parellopiped) is shown in Figure 6.19a in reference to a global Cartesian coordinate system. Here, we utilize the natural coordinates r, s, t of Figure 6.19b, defined as

$$r = \frac{x - \bar{x}}{a}$$
$$s = \frac{y - \bar{y}}{b} \tag{6.67}$$
$$t = \frac{z - \bar{z}}{c}$$

where $2a$, $2b$, $2c$ are the dimensions of the element in the x, y, z coordinates, respectively, and the coordinates of the element centroid are

$$\bar{x} = \frac{x_2 - x_1}{2}$$
$$\bar{y} = \frac{y_3 - y_2}{2} \tag{6.68}$$
$$\bar{z} = \frac{z_5 - z_1}{2}$$

The natural coordinates are defined such that the coordinate values vary between -1 and $+1$ over the domain of the element. As with the plane rectangular element, the natural coordinates provide for a straightforward development of the interpolation functions by using the appropriate monomial terms to satisfy nodal conditions. As we illustrated the procedure in several previous developments, we do not repeat the details here. Instead, we simply write the interpolation functions in terms of the natural coordinates and request that the reader

verify satisfaction of all nodal conditions. The interpolation functions are

$$\begin{aligned}
N_1 &= \frac{1}{8}(1-r)(1-s)(1+t)\\
N_2 &= \frac{1}{8}(1+r)(1-s)(1+t)\\
N_3 &= \frac{1}{8}(1+r)(1+s)(1+t)\\
N_4 &= \frac{1}{8}(1-r)(1+s)(1+t)\\
N_5 &= \frac{1}{8}(1-r)(1-s)(1-t)\\
N_6 &= \frac{1}{8}(1+r)(1-s)(1-t)\\
N_7 &= \frac{1}{8}(1+r)(1+s)(1-t)\\
N_8 &= \frac{1}{8}(1-r)(1+s)(1-t)
\end{aligned} \tag{6.69}$$

and the field variable is described as

$$\phi(x, y, z) = \sum_{i=1}^{8} N_i(r, s, t)\phi_i \tag{6.70}$$

If Equation 6.70 is expressed in terms of the global Cartesian coordinates, it is found to be of the form

$$\phi(x, y, z) = a_0 + a_1x + a_2y + a_3z + a_4xy + a_5xz + a_6yz + a_7xyz \tag{6.71}$$

showing that the field variable is expressed as an incomplete, symmetric polynomial. Geometric isotropy is therefore assured. The compatibility requirement is satisfied, as is the completeness condition. Recall that completeness requires that the first partial derivatives must be capable of assuming constant values (for C^0 problems). If, for example, we take the first partial derivative of Equation 6.71 with respect to x, we obtain

$$\frac{\partial\phi}{\partial x} = a_1 + a_4y + a_5z + a_7yz \tag{6.72}$$

which certainly does not appear to be constant at first glance. However, if we apply the derivative operation to Equation 6.70 while noting that

$$\frac{\partial\phi}{\partial x} = \frac{1}{a}\frac{\partial\phi}{\partial r} \tag{6.73}$$

the result is

$$\frac{\partial \phi}{\partial x} = \frac{1}{8a}(1-s)(1+t)(\phi_2 - \phi_1) + \frac{1}{8a}(1+s)(1+t)(\phi_3 - \phi_4)$$
$$+ \frac{1}{8a}(1-s)(1-t)(\phi_6 - \phi_5) + \frac{1}{8a}(1+s)(1-t)(\phi_7 - \phi_8) \qquad (6.74)$$

Referring to Figure 6.19, observe that, if the gradient of the field variable in the x direction is constant, $\partial\phi/\partial x = C$, the nodal values are related by

$$\begin{aligned}
\phi_2 &= \phi_1 + \frac{\partial \phi}{\partial x}\,dx = \phi_1 + C(2a) \\
\phi_3 &= \phi_4 + \frac{\partial \phi}{\partial x}\,dx = \phi_4 + C(2a) \\
\phi_6 &= \phi_5 + \frac{\partial \phi}{\partial x}\,dx = \phi_5 + C(2a) \\
\phi_7 &= \phi_8 + \frac{\partial \phi}{\partial x}\,dx = \phi_8 + C(2a)
\end{aligned} \qquad (6.75)$$

Substituting these relations into Equation 6.74, we find

$$\frac{\partial \phi}{\partial x} = \frac{1}{8a}[(1-s)(1+t)(2aC) + (1+s)(1+t)(2aC)$$
$$+ (1-s)(1+t)(2aC) + (1+s)(1-t)(2aC)] \qquad (6.76a)$$

which, on expansion and simplification, results in

$$\frac{\partial \phi}{\partial x} \equiv C \qquad (6.76b)$$

Observing that this result is valid at any point (r, s, t) within the element, it follows that the specified interpolation functions indeed allow for a constant gradient in the x direction. Following similar procedures shows that the other partial derivatives also satisfy the completeness condition.

6.8 ISOPARAMETRIC FORMULATION

The finite element method is a powerful technique for analyzing engineering problems involving complex, irregular geometries. However, the two- and three-dimensional elements discussed so far in this chapter (triangle, rectangle, tetrahedron, brick) cannot always be efficiently used for irregular geometries. Consider the plane area shown in Figure 6.20a, which is to be discretized via a mesh of finite elements. A possible mesh using triangular elements is shown in Figure 6.20b. Note that the outermost "row" of elements provides a chordal approximation to the circular boundary, and as the size of the elements is decreased (and the number of elements increased), the approximation becomes increasingly

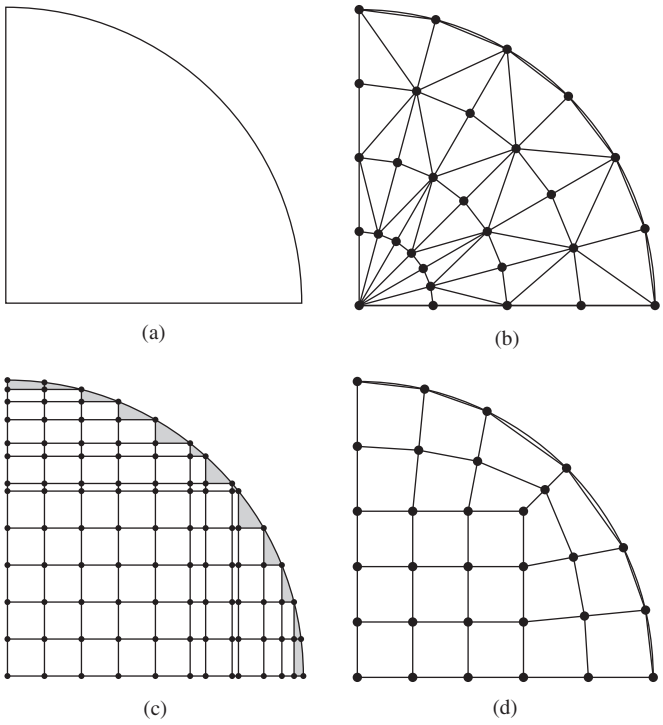

Figure 6.20
(a) A domain to be modeled. (b) Triangular elements. (c) Rectangular elements. (d) Rectangular and quadrilateral elements.

closer to the actual geometry. However, also note that the elements in the inner "rows" become increasingly slender (i.e., the height to base ratio is large). In general, the ratio of the largest characteristic dimension of an element to the smallest characteristic dimension is known as the *aspect ratio*. Large aspect ratios increase the inaccuracy of the finite element representation and have a detrimental effect on convergence of finite element solutions [8]. An aspect ratio of 1 is ideal but cannot always be maintained. (Commercial finite element software packages provide warnings when an element's aspect ratio exceeds some predetermined limit.) In Figure 6.20b, to maintain a reasonable aspect ratio for the inner elements, it would be necessary to reduce the height of each row of elements as the center of the sector is approached. This observation is also in keeping with the convergence requirements of the *h*-refinement method. Although the triangular element can be used to closely approximate a curved boundary, other considerations dictate a relatively large number of elements and associated computation time.

If we consider rectangular elements as in Figure 6.20c (an intentionally crude mesh for illustrative purposes), the problems are apparent. Unless the elements are very small, the area of the domain excluded from the model (the

shaded area in the figure) may be significant. For the case depicted, a large number of very small square elements best approximates the geometry.

At this point, the astute reader may think, Why not use triangular and rectangular elements in the same mesh to improve the model? Indeed, a combination of the element types can be used to improve the geometric accuracy of the model. The shaded areas of Figure 6.20c could be modeled by three-node triangular elements. Such combination of element types may not be the best in terms of solution accuracy since the rectangular element and the triangular element have, by necessity, different order polynomial representations of the field variable. The field variable *is* continuous across such element boundaries; this is guaranteed by the finite element formulation. However, conditions on derivatives of the field variable for the two element types are quite different. On a curved boundary such as that shown, the triangular element used to fill the "gaps" left by the rectangular elements may also have adverse aspect ratio characteristics.

Now examine Figure 6.20d, which shows the same area meshed with rectangular elements and a new element applied near the periphery of the domain. The new element has four nodes, straight sides, but is not rectangular. (Please note that the mesh shown is intentionally coarse for purposes of illustration.) The new element is known as a general two-dimensional *quadrilateral element* and is seen to mesh ideally with the rectangular element as well as approximate the curved boundary, just like the triangular element. The four-node quadrilateral element is derived from the four-node rectangular element (known as the *parent* element) element via a *mapping* process. Figure 6.21 shows the parent element and its natural (r, s) coordinates and the quadrilateral element in a global Cartesian coordinate system. The geometry of the quadrilateral element is described by

$$x = \sum_{i=1}^{4} G_i(x, y)x_i \tag{6.77}$$

$$y = \sum_{i=1}^{4} G_i(x, y)y_i \tag{6.78}$$

where the $G_i(x, y)$ can be considered as *geometric* interpolation functions, and each such function is associated with a particular node of the quadrilateral

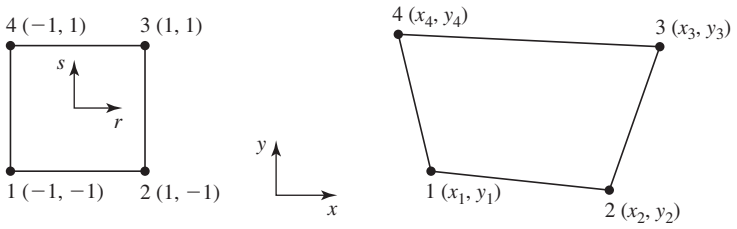

Figure 6.21 Mapping of a parent element into an isoparametric element. A rectangle is shown for example.

element. Given the geometry and the form of Equations 6.77 and 6.78, each function $G_i(x, y)$ must evaluate to unity at its associated node and to zero at each of the other three nodes.

These conditions are exactly the same as those imposed on the interpolation functions of the parent element. Consequently, the interpolation functions for the parent element can be used for the geometric functions, if we map the coordinates so that

$$\begin{aligned}(r, s) &= (-1, -1) \Rightarrow (x_1, y_1)\\ (r, s) &= (1, -1) \Rightarrow (x_2, y_2)\\ (r, s) &= (1, 1) \Rightarrow (x_3, y_3)\\ (r, s) &= (-1, 1) \Rightarrow (x_4, y_4)\end{aligned} \tag{6.79}$$

where the symbol $\Rightarrow$ is read as "maps to" or "corresponds to." Note that the (r, s) coordinates used here are *not* the same as those defined by Equation 6.54. Instead, these are the actual rectangular coordinates of the 2 unit by 2 unit parent element.

Consequently, the geometric expressions become

$$\begin{aligned}x &= \sum_{i=1}^{4} N_i(r, s)x_i\\ y &= \sum_{i=1}^{4} N_i(r, s)y_i\end{aligned} \tag{6.80}$$

Clearly, we can also express the field variable variation in the quadrilateral element as

$$\phi(x, y) = \phi(r, s) = \sum_{i=1}^{4} N_i(r, s)\phi_i \tag{6.81}$$

if the mapping of Equation 6.79 is used, since all required nodal conditions are satisfied. Since the same interpolation functions are used for both the field variable and description of element geometry, the procedure is known as *isoparametric* (constant parameter) *mapping*. The element defined by such a procedure is known as an *isoparametric element.* The mapping of element boundaries is illustrated in the following example.

EXAMPLE 6.3

Figure 6.22 shows a quadrilateral element in global coordinates. Show that the mapping described by Equation 6.80 correctly describes the line connecting nodes 2 and 3 and determine the (x, y) coordinates corresponding to $(r, s) = (1, 0.5)$

■ Solution

First, we determine the equation of the line passing through nodes 2 and 3 strictly by geometry, using the equation of a two-dimensional straight line $y = mx + b$. Using the

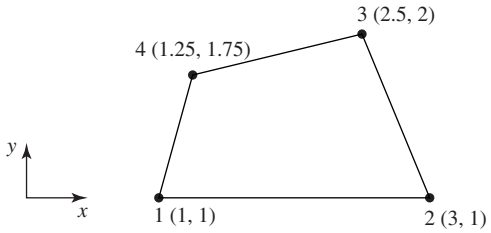

Figure 6.22 Quadrilateral element for Example 6.3.

known coordinates of nodes 2 and 3, we have

$$\text{Node 2:} \qquad 1 = 3m + b$$

$$\text{Node 3:} \qquad 2 = 2.5m + b$$

Solving simultaneously, the slope is

$$m = -2$$

and the y intercept is

$$b = 7$$

Therefore, element edge 2-3 is described by

$$y = -2x + 7$$

Using the interpolation functions given in Equation 6.56 and substituting nodal x and y coordinates, the geometric mapping of Equation 6.80 becomes

$$x = \frac{1}{4}(1-r)(1-s)(1) + \frac{1}{4}(1+r)(1-s)(3) + \frac{1}{4}(1+r)(1+s)(2.5) + \frac{1}{4}(1-r)(1+s)(1.25)$$

$$y = \frac{1}{4}(1-r)(1-s)(1) + \frac{1}{4}(1+r)(1-s)(1) + \frac{1}{4}(1+r)(1+s)(2) + \frac{1}{4}(1-r)(1+s)(1.75)$$

Noting that edge 2-3 corresponds to $r = 1$, the last two equations become

$$x = \frac{3}{2}(1-s) + \frac{2.5}{2}(1+s) = \frac{5.5}{2} - \frac{0.5}{2}s$$

$$y = \frac{1}{2}(1-s) + (1+s) = \frac{3}{2} + \frac{1}{2}s$$

Eliminating s gives

$$2x + y = \frac{14}{2}$$

which is the same as

$$y = -2x + 7$$

as desired.

For $(r, s) = (1, 0.5)$, we obtain

$$x = \frac{5.5}{2} - \frac{0.5}{2}(0.5) = 2.625$$

$$y = \frac{3}{2} + \frac{1}{2}(0.5) = 1.75$$

In formulating element characteristic matrices, various derivatives of the interpolation functions with respect to the global coordinates are required, as previously demonstrated. In isoparametric elements, both element geometry and variation of the interpolation functions are expressed in terms of the natural coordinates of the parent element, so some additional mathematical complication arises. Specifically, we must compute $\partial N_i/\partial x$ and $\partial N_i/\partial y$ (and, possibly, higher-order derivatives). Since the interpolation functions are expressed in (r, s) coordinates, we can formally write these derivatives as

$$\begin{aligned} \frac{\partial N_i}{\partial x} &= \frac{\partial N_i}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial N_i}{\partial s}\frac{\partial s}{\partial x} \\ \frac{\partial N_i}{\partial y} &= \frac{\partial N_i}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial N_i}{\partial s}\frac{\partial s}{\partial y} \end{aligned} \tag{6.82}$$

However, unless we invert the relations in Equation 6.80, the partial derivatives of the natural coordinates with respect to the global coordinates are not known. As it is virtually impossible to invert Equation 6.80 to explicit algebraic expressions, a different approach must be taken.

We take an indirect approach, by first examining the partial derivatives of the field variable with respect to the natural coordinates. From Equation 6.81, the partial derivatives of the field variable with respect to the natural coordinates can be expressed formally as

$$\begin{aligned} \frac{\partial \phi}{\partial r} &= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial r} \\ \frac{\partial \phi}{\partial s} &= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial s} \end{aligned} \tag{6.83}$$

In light of Equation 6.81, computation of the partial derivatives of the field variable requires the partial derivatives of each interpolation function as

$$\begin{aligned} \frac{\partial N_i}{\partial r} &= \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial r} \\ \frac{\partial N_i}{\partial s} &= \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial s} \end{aligned} \qquad i = 1, 4 \tag{6.84}$$

Writing Equation 6.84 in matrix form,

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial r} \\ \dfrac{\partial N_i}{\partial s} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix} \qquad i = 1, 4 \tag{6.85}$$

we observe that the 2×1 vector on the left-hand side is known, since the interpolation functions are expressed explicitly in the natural coordinates. Similarly, the terms in the 2×2 coefficient matrix on the right-hand side are known via Equation 6.80. The latter, known as the *Jacobian matrix,* denoted [J], is given by

$$[\mathrm{J}] = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{bmatrix} = \begin{bmatrix} \displaystyle\sum_{i=1}^{4} \frac{\partial N_i}{\partial r} x_i & \displaystyle\sum_{i=1}^{4} \frac{\partial N_i}{\partial r} y_i \\ \displaystyle\sum_{i=1}^{4} \frac{\partial N_i}{\partial s} x_i & \displaystyle\sum_{i=1}^{4} \frac{\partial N_i}{\partial s} y_i \end{bmatrix} \tag{6.86}$$

If the inverse of the Jacobian matrix can be determined, Equation 6.85 can be solved for the partial derivatives of the interpolation functions with respect to the global coordinates to obtain

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix} = [\mathrm{J}]^{-1} \begin{Bmatrix} \dfrac{\partial N_i}{\partial r} \\ \dfrac{\partial N_i}{\partial s} \end{Bmatrix} = \begin{bmatrix} \mathrm{I}_{11} & \mathrm{I}_{12} \\ \mathrm{I}_{21} & \mathrm{I}_{22} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_i}{\partial r} \\ \dfrac{\partial N_i}{\partial s} \end{Bmatrix} \qquad i = 1, 4 \tag{6.87}$$

with the terms of the inverse of the Jacobian matrix denoted I_{ij} for convenience. Equation 6.87 can be used to obtain the partial derivatives of the field variable with respect to the global coordinates, as required in discretizing a governing differential equation by the finite element method. In addition, the derivatives are required in computing the "secondary" variables, including strain (then stress) in structural problems and heat flux in heat transfer. These and other problems are illustrated in subsequent chapters.

As we also know, various integrations are required to obtain element stiffness matrices and load vectors. For example, in computing the terms of the conductance matrix for two-dimensional heat transfer elements, integrals of the form

$$\iint_A \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} \right) \mathrm{d}A$$

are encountered, and the integration is to be performed over the area of the element in global coordinates. However, for an isoparametric element such as the quadrilateral being discussed, the interpolation functions are in terms of the parent element coordinates. Hence, it is necessary to transform such integrals to

the natural coordinates. From Equation 6.87, we have

$$\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} = \left(\mathrm{I}_{11}\frac{\partial N_i}{\partial r} + \mathrm{I}_{12}\frac{\partial N_i}{\partial s}\right)\left(\mathrm{I}_{11}\frac{\partial N_j}{\partial r} + \mathrm{I}_{12}\frac{\partial N_j}{\partial s}\right) \tag{6.88}$$

so the integrand is transformed using the terms of $[\mathrm{J}]^{-1}$. As shown in advanced calculus [9], the differential area relationship is

$$\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = |\mathrm{J}|\,\mathrm{d}r\,\mathrm{d}s \tag{6.89}$$

so integrals of the form described previously become

$$\iint_A \left(\frac{\partial N_i}{\partial x}\,\frac{\partial N_j}{\partial x}\right)\mathrm{d}A = \int_{-1}^{1}\int_{-1}^{1} \left(\mathrm{I}_{11}\frac{\partial N_i}{\partial r} + \mathrm{I}_{12}\frac{\partial N_i}{\partial s}\right)\left(\mathrm{I}_{11}\frac{\partial N_j}{\partial r} + \mathrm{I}_{12}\frac{\partial N_j}{\partial s}\right)|\mathrm{J}|\,\mathrm{d}r\,\mathrm{d}s \tag{6.90}$$

Such integrals are discussed in greater detail in later chapters in problem-specific contexts. The intent of this discussion is to emphasize the importance of the Jacobian matrix in development of isoparametric elements.

Rather than work with individual interpolation functions, it is convenient to combine Equations 6.84 and 6.85 into matrix form as

$$\begin{Bmatrix} \dfrac{\partial [N]}{\partial r} \\ \dfrac{\partial [N]}{\partial s} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial [N]}{\partial x} \\ \dfrac{\partial [N]}{\partial y} \end{Bmatrix} \tag{6.91}$$

where $[N]$ is the 1×4 row matrix

$$[N] = [N_1 \quad N_2 \quad N_3 \quad N_4] \tag{6.92}$$

and Equation 6.91 in matrix notation is the same as

$$\begin{Bmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{\partial}{\partial s} \end{Bmatrix} [N] = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial s} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial s} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} [N] \tag{6.93}$$

We use this matrix notation to advantage in later chapters, when we examine specific applications.

While the isoparametric formulation just described is mathematically straightforward, the algebraic complexity is significant, as illustrated in the following example.

EXAMPLE 6.4

Determine the Jacobian matrix for a four-node, two-dimensional quadrilateral element having the parent element whose interpolation functions are given by Equation 6.56.

■ **Solution**

The partial derivatives of x and y with respect to r and s per Equations 6.56 and 6.80 are

$$\frac{\partial x}{\partial r} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial r} x_i = \frac{1}{4}[-(1-s)x_1 + (1-s)x_2 + (1+s)x_3 - (1+s)x_4]$$

$$\frac{\partial y}{\partial r} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial r} y_i = \frac{1}{4}[-(1-s)y_1 + (1-s)y_2 + (1+s)y_3 - (1+s)y_4]$$

$$\frac{\partial x}{\partial s} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial s} x_i = \frac{1}{4}[-(1-r)x_1 - (1+r)x_2 + (1+r)x_3 + (1-r)x_4]$$

$$\frac{\partial y}{\partial s} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial s} y_i = \frac{1}{4}[-(1-r)y_1 - (1+r)y_2 + (1+r)y_3 + (1-r)y_4]$$

The Jacobian matrix is then

$$[\mathrm{J}] = \frac{1}{4}\begin{bmatrix} (1-s)(x_2-x_1) + (1+s)(x_3-x_4) & (1-s)(y_2-y_1) + (1+s)(y_3-y_4) \\ (1-r)(x_4-x_1) + (1+r)(x_3-x_2) & (1-r)(y_4-y_1) + (1+r)(y_3-y_2) \end{bmatrix}$$

Note that finding the inverse of this Jacobian matrix in explicit form is not an enviable task. The task is impossible except in certain special cases. For this reason, isoparametric element formulation is carried out using numerical integration, as discussed in Section 6.10.

The isoparametric formulation is by no means limited to linear parent elements. Many higher-order isoparametric elements have been formulated and used successfully [1]. Figure 6.23 depicts the isoparametric elements corresponding to the six-node triangle and the eight-node rectangle. Owing to the mapping being described by quadratic functions of the parent elements, the resulting elements have curved boundaries, which are also described by quadratic functions of the global coordinates. Such elements can be used to closely approximate irregular boundaries. However, note that curved elements do not, in general, exactly match a specified boundary curve.

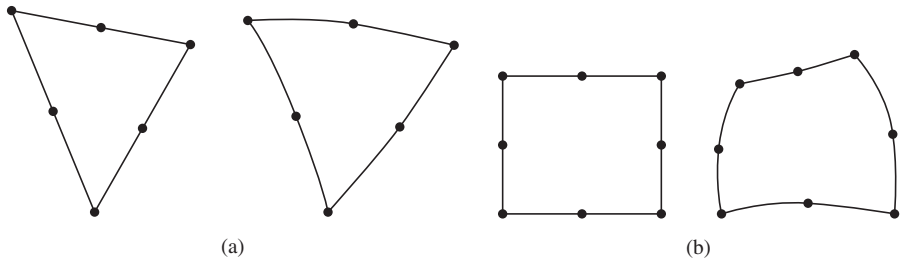

Figure 6.23 Isoparametric mapping of quadratic elements into curved elements: (a) Six-node triangle. (b) Eight-node rectangle.

6.9 AXISYMMETRIC ELEMENTS

Many three-dimensional field problems in engineering exhibit symmetry about an axis of rotation. Such problems, known as *axisymmetric problems,* can be solved using two-dimensional finite elements, which are most conveniently described in cylindrical (r, θ, z) coordinates. The required conditions for a problem to be axisymmetric are as follows:

1. The problem domain must possess an axis of symmetry, which is conventionally taken as the z axis; that is, the domain is geometrically a solid of revolution.
2. The boundary conditions are symmetric about the axis of revolution; thus, all boundary conditions are independent of the circumferential coordinate θ.
3. All loading conditions are symmetric about the axis of revolution; thus, they are also independent of the circumferential coordinate.

In addition, the material properties must be symmetric about the axis of revolution. This condition is, of course, automatically satisfied for isotropic materials.

If these conditions are met, the field variable ϕ is a function of radial and axial (r, z) coordinates only and described mathematically by two-dimensional governing equations.

Figure 6.24a depicts a cross section of an axisymmetric body assumed to be the domain of an axisymmetric problem. The cross section could represent the wall of a pressure vessel for stress or heat transfer analysis, an annular region of fluid flow, or blast furnace for steel production, to name a few examples. In

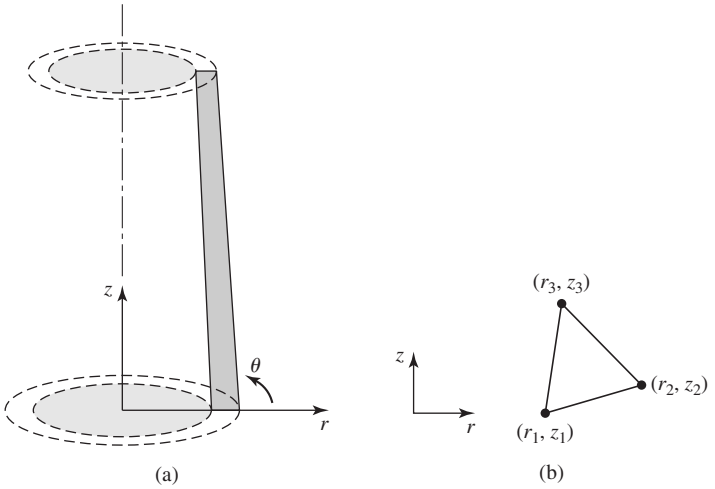

Figure 6.24
(a) An axisymmetric body and cylindrical coordinates. (b) A three-node triangle in cylindrical coordinates at an arbitrary value θ.

Figure 6.24b, a three-node triangular element is shown having nodal coordinates (r_i, z_i). In the axisymmetric case, the field variable is discretized as

$$\phi(r, z) = \sum_{i=1}^{3} N_i(r, z)\phi_i \tag{6.94}$$

where the interpolation functions $N_i(r, z)$ must satisfy the usual nodal conditions.

Noting that the nodal conditions are satisfied by the interpolation functions defined by Equation 6.37 if we simply substitute r for x and z for y, the interpolation functions for the axisymmetric triangular element are immediately obtained. Similarly, the interpolation functions in terms of area coordinates are also applicable.

Since, by definition of an axisymmetric problem, the problem, therefore its solution, is independent of the circumferential coordinate θ, so must be the interpolation functions. Consequently, any two-dimensional element and associated interpolation functions can be used for axisymmetric elements. What is the difference? The axisymmetric element is physically three dimensional. As depicted in Figure 6.25, the triangular axisymmetric element is actually a prism of revolution. The "nodes" are circles about the axis of revolution of the body, and the nodal conditions are satisfied at every point along the circumference defined by the node of a two-dimensional element. Although we use a triangular element for illustration, we reiterate that any two-dimensional element can be used to formulate an axisymmetric element.

As is shown in subsequent chapters in terms of specific axisymmetric problems, integration of various functions of the interpolation functions over the volume are required for element formulation. Symbolically, such integrals are represented as

$$F(r, \theta, z) = \iiint_V f(r, \theta, z)\, dV = \iiint f(r, \theta, z) r\, dr\, d\theta\, dz \tag{6.95}$$

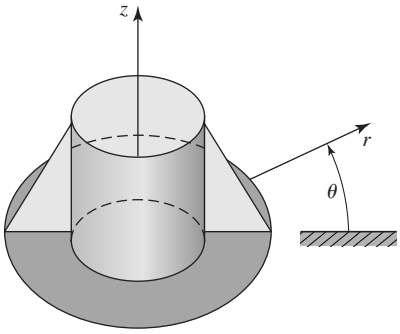

Figure 6.25 A three-dimensional representation of an axisymmetric element based on a three-node triangular element.

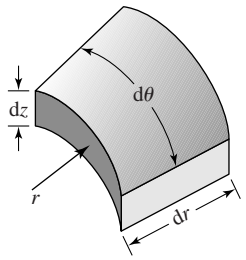

Figure 6.26 Differential volume in cylindrical coordinates.

where V is the volume of an element and $dV = r\,dr\,d\theta\,dz$ is the differential volume depicted in Figure 6.26. For axial symmetry, the integrand is independent of the circumferential coordinate θ, so the integration indicated in Equation 6.95 becomes

$$F(r, \theta, z) = F(r, z) = 2\pi \iint_A f(r, z) r\,dr\,dz \tag{6.96}$$

Equation 6.96 shows that the integration operations required for formulation of axisymmetric elements are distinctly different from those of two-dimensional elements, even though the interpolation functions are essentially identical. As stated, we show applications of axisymmetric elements in subsequent chapters. Also, *any* two-dimensional element can be readily converted to an axisymmetric element, provided the true three-dimensional nature of the element is taken into account when element characteristic matrices are formulated.

EXAMPLE 6.5

In following chapters, we show that integrals of the form

$$\int_V N_i N_j\,dV$$

where N_i, N_j are interpolation functions and V is element volume, must be evaluated in formulation of element matrices. Evaluate the integral with $i = 1$, $j = 2$ for an axisymmetric element based on the three-node triangle using the area coordinates as the interpolation functions.

■ Solution

For the axisymmetric element, we use Equation 6.96 to write

$$\int_V N_i N_j\,dV = 2\pi \int_A N_i N_j r\,dr\,dz = 2\pi \int_A L_i L_j r\,dr\,dz$$

where A is the element area. Owing to the presence of the variable r in the integrand, the integration formula, Equation 6.49, cannot be applied directly. However, we can express r in terms of the nodal coordinates r_1, r_2, r_3 and the area coordinates as

$$r = L_1 r_1 + L_2 r_2 + L_3 r_3$$

Then, noting that $\mathrm{d}r\,\mathrm{d}z = \mathrm{d}A$, we have

$$2\pi \int_A L_i L_j r \,\mathrm{d}r\,\mathrm{d}z = 2\pi \int_A L_i L_j (L_1 r_1 + L_2 r_2 + L_3 r_3)\,\mathrm{d}A$$

which is of the appropriate form for application of the integration formula. For $i = 1$, $j = 2$, the integral becomes

$$\begin{aligned} 2\pi \int_A L_1 L_2 r \,\mathrm{d}r\,\mathrm{d}z &= 2\pi \int_A L_1 L_2 (L_1 r_1 + L_2 r_2 + L_3 r_3)\,\mathrm{d}A \\ &= 2\pi r_1 \int_A L_1^2 L_2 \,\mathrm{d}A + 2\pi r_2 \int_A L_1 L_2^2 \,\mathrm{d}A + 2\pi r_3 \int_A L_1 L_2 L_3 \,\mathrm{d}A \end{aligned}$$

Applying the integration formula to each of the three integrals on the right,

$$\begin{aligned} &2\pi \int_A L_1 L_2 r \,\mathrm{d}r\,\mathrm{d}z \\ &\quad = 4\pi A \left[r_1 \frac{(2!)(1!)(0!)}{(2+1+0+2)!} + r_2 \frac{(1!)(2!)(0!)}{(1+2+0+2)!} + r_3 \frac{(1!)(1!)(1!)}{(1+1+1+2)!} \right] \\ &\quad = 4\pi A \left(\frac{2r_1}{120} + \frac{2r_2}{120} + \frac{r_3}{120} \right) = \frac{\pi A}{30}(2r_1 + 2r_2 + r_3) \end{aligned}$$

The integration technique used in Example 6.5 is also applicable to higher-order, straight-sided triangular elements, as shown in the next example.

EXAMPLE 6.6

For an axisymmetric element based on the six-node, quadratic triangular element having interpolation functions given by Equation 6.47, evaluate the integral

$$I = \int_V N_2 N_4 \,\mathrm{d}V$$

■ Solution

Using Equation 6.93,

$$I = \int_V N_2 N_4 \,\mathrm{d}V = 2\pi \int_A N_2 N_4 r \,\mathrm{d}r\,\mathrm{d}z = 2\pi \int_A L_2(2L_2 - 1)(4L_1 L_2) r \,\mathrm{d}r\,\mathrm{d}z$$

Now observe that, even though the interpolation functions vary quadratically over the element area, the area coordinates, by definition, vary linearly. Since the element sides are straight, the radial coordinate can still be expressed as

$$r = L_1 r_1 + L_2 r_2 + L_3 r_3$$

Therefore, we have

$$I = 2\pi \int_A L_2(2L_2 - 1)(4L_1 L_2)(L_1 r_1 + L_2 r_2 + L_3 r_3)\,\mathrm{d}A$$

or

$$I = 16\pi \int_A \left(L_1^2 L_2^3 r_1 + L_1 L_2^4 r_2 + L_1 L_2^3 L_3 r_3\right) \mathrm{d}A$$
$$- 8\pi \int_A \left(L_1^2 L_2^2 r_1 + L_1 L_2^3 r_2 + L_1 L_2^2 L_3 r_3\right) \mathrm{d}A$$

Application of the integration formula, Equation 6.49, to each of the six integrals represented here (left as an exercise), we find

$$I = \frac{\pi A}{315}(6r_2 - 4r_1 - 2r_3)$$

6.10 NUMERICAL INTEGRATION: GAUSSIAN QUADRATURE

Previous chapters show that integration of various functions of the field variable are required for formulation of finite element characteristic matrices. Chapter 5 reveals that the Galerkin method requires integration over the element domain (and, as seen, physical volume), once for each interpolation function (trial solution). In fact, an integration is required to obtain the value of every component of the stiffness matrix of a finite element. In addition, integrations are required to obtain nodal equivalents of nonnodal loadings.

In this chapter, we focus primarily on polynomial representations of the discretized representations of the field variable. In subsequent formulation of element characteristic matrices, we are faced with integrations of polynomial forms. A simple polynomial is relatively easy to integrate in closed form. In many cases, however, the integrands are rational functions, that is, ratios of polynomials; and these are quite tedious to integrate directly. In either case, in the finite element context, where large numbers of elements, hence huge numbers of integrations, are required, analytical methods are not efficient. Finite element software packages do not incorporate explicit integration of the element formulation equations. Instead, they use numerical techniques, the most popular of which is *Gaussian* (or *Gauss-Legendre*) *quadrature* [10].

The concept of Gaussian quadrature is first illustrated in one dimension in the context of an integral of the form

$$I = \int_{x_1}^{x_2} h(x)\,\mathrm{d}x \tag{6.97}$$

Via the change of variable $r = ax + b$, Equation 6.97 can be converted to

$$I = \int_{-1}^{1} f(r)\,\mathrm{d}r \tag{6.98}$$

with $dr = a\,dx$. The coefficients a and b are determined so that the integration limits become minus and plus unity. This is conventional for the numerical integration procedure and also accords nicely with the range of the natural coordinates of many of the elements discussed in this chapter.

Per the Gaussian integration procedure, the integration represented by Equation 6.98 can be approximated by

$$I = \sum_{i=1}^{m} W_i f(r_i) \tag{6.99}$$

where W_i are Gaussian weighting factors and r_i are known as *sampling points* or *Gauss points*. The weighting factors and sampling points are determined [9] to minimize error, particularly in terms of polynomial functions. Of particular import in finite element analysis, a polynomial of order n can be *exactly* integrated. Referring to Equation 6.99, use of m sampling points and weighting factors results in an exact value of the integral for a polynomial of order $2m - 1$, if the sampling points and weighting factors are chosen in accordance with Table 6.1. This means, for example, that a cubic polynomial can be exactly integrated by Equation 6.99, using only two sampling points and evaluating the integrand at those points, multiplying by the weighting factors, and summing the results.

To illustrate how the sampling points and weighting factors are determined, we formally integrate a general polynomial in one dimension as

$$\int_{-1}^{1} (a_0 + a_1 r + a_2 r^2 + a_3 r^3 + \cdots + a_n r^n)\,dr = 2a_0 + \frac{2}{3}a_2 + \frac{2}{5}a_4 + \cdots + \frac{2}{n+1}a_n \tag{6.100}$$

Table 6.1 Sampling points and weighting factors for Gaussian quadrature numerical integration of $\int_{-1}^{1} f(r)\,dr \approx \sum_{i=1}^{m} W_i f(r_i)$. This is an abridged table, giving values sufficient for exact integration of a polynomial of order seven or less

m	r_i	W_i
1	0.0	2.0
2	0.577350269189626. . .	1.0
	−0.577350269189626. . .	1.0
3	0.0	0.888888888888889
	0.774596669241483	0.555555555555556
	−0.774596669241483	0.555555555555556
4	0.339981043583856	0.652145154862526
	−0.339981043583856	0.652145154862526
	0.861136311590453	0.347854845137454
	−0.861136311590453	0.347854845137454

and we observe that, owing to the symmetry of the integration limits, all odd powers integrate to zero. Also note that we assume here that n is an even integer. The approximation to the integral per Equation 6.99 is

$$\begin{aligned} I \approx \sum_{i=1}^{m} W_i f(r_i) = {} & W_1\left(a_0 + a_1 r_1 + a_2 r_1^2 + a_3 r_1^3 + \cdots + a_n r_1^n\right) \\ & + W_2\left(a_0 + a_1 r_2 + a_2 r_2^2 + a_3 r_2^3 + \cdots + a_n r_2^n\right) \\ & + W_3\left(a_0 + a_1 r_3 + a_2 r_3^2 + a_3 r_3^3 + \cdots + a_n r_3^n\right) \\ & \vdots \\ & + W_m\left(a_0 + a_1 r_m + a_2 r_m^2 + a_3 r_m^3 + \cdots + a_n r_m^n\right) \end{aligned} \tag{6.101}$$

Comparing Equations 6.100 and 6.101 in terms of the coefficients a_j of the polynomial, the approximation of Equation 6.101 becomes exact if

$$\begin{aligned} \sum_{i=1}^{m} W_i &= 2 \\ \sum_{i=1}^{m} W_i r_i &= 0 \\ \sum_{i=1}^{m} W_i r_i^2 &= \frac{2}{3} \\ \sum_{i=1}^{m} W_i r_i^3 &= 0 \\ \sum_{i=1}^{m} W_i r_i^4 &= \frac{2}{5} \\ &\vdots \\ \sum_{i=1}^{m} W_i r_i^{n-1} &= 0 \\ \sum_{i=1}^{m} W_i r_i^n &= \frac{2}{n+1} \end{aligned} \tag{6.102}$$

where m is the number of sampling (Gauss) integration points.

Equation 6.102 represents n equations in $2m + 1$ unknowns. The unknowns are the weighting factors W_i, the sampling point values r_i, and most troublesome, the *number* of sampling points m. While we do not go into the complete theory of Gaussian quadrature, we illustrate by example how the sampling points and weights can be determined using both the equations and logic. First, note that the equations corresponding to odd powers of the polynomial indicate a zero summation. Second, note that the first equation is applicable regardless of the order of the polynomial; that is, the weighting factors must sum to the value of 2 if exactness is to be achieved.

If, for example, we have a linear polynomial $n = 1$, the first two of Equation 6.102 are applicable and lead to the conclusions that we need only one sampling point and that the appropriate values of the weighting factor and sample point to satisfy the two equations (in this case) are $W_1 = 2$ and $r_1 = 0$. Next, consider the case of a cubic polynomial, $n = 3$. In this case, we have

$$\sum_{i=1}^{m} W_i = 2$$

$$\sum_{i=1}^{m} W_i r_i = 0$$

$$\sum_{i=1}^{m} W_i r_i^2 = \frac{2}{3}$$

representing three equations in $2m + 1$ unknowns. If we let $m = 1$, the first two equations lead to $W_1 = 2$, $r_1 = 0$, but the third equation cannot be satisfied. On the other hand, if $m = 2$, we have

$$W_1 + W_2 = 2$$

$$W_1 r_1 + W_2 r_2 = 0$$

$$W_1 r_1^2 + W_2 r_2^2 = \frac{2}{3}$$

a system of three equations in four unknowns. We cannot directly solve these equations, but if we examine the case $W_1 = W_2 = 1$ and $r_1 = -r_2$, the first two equations are satisfied and the third equation becomes

$$r_1^2 + r_2^2 = 2r_1^2 = \frac{2}{3} \Rightarrow r_1 = \sqrt{\frac{1}{3}} = \frac{\sqrt{3}}{3} = 0.57735\ldots$$

corresponding exactly to the second entry in Table 6.1. These weighting factors and Gauss points also integrate a quadratic polynomial exactly. The reader is urged to note that, because of the zero result from integrating the odd powers in the polynomial, exact results are obtained for *two* polynomial orders for each set of sampling points and weighting factors.

This discussion is by no means intended to be mathematically rigorous in terms of the theory underlying numerical integration. The intent is to give some insight as to the rationale behind the numerical values presented in Table 6.1.

EXAMPLE 6.7

Evaluate the integral

$$f(r) = \int_{-1}^{1} (r^2 - 3r + 7)\, \mathrm{d}r$$

using Gaussian quadrature so that the result is exact.

■ **Solution**

As the integrand is a polynomial of order 2, we have, for exact integration, $2m - 1 = 2$, which results in the required number of sampling points as $m = 3/2$. The calculated number of sampling points must be rounded *up* to the nearest integer value, so in this case, we must use two sampling points. Per Table 6.1, the sampling points are $r_i = \pm 0.5773503$ and the weighting factors are $W_i = 1.0$, $i = 1, 2$. Therefore,

$$\int_{-1}^{1} (r^2 - 3r + 7)\,\mathrm{d}r = (1)[(0.5773503)^2 - 3(0.5773503) + 7] + (1)[(-0.5773503)^2 - 3(-0.5773503) + 7]$$

$$\int_{-1}^{1} (r^2 - 3r + 7)\,\mathrm{d}r = 14.666667$$

The result is readily verified as, indeed, being exact by direct integration.

The Gaussian quadrature numerical integration procedure is by no means limited to one dimension. In finite element analysis, integrals of the forms

$$I = \int_{-1}^{1}\int_{-1}^{1} f(r, s)\,\mathrm{d}r\,\mathrm{d}s$$
$$I = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(r, s, t)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t \tag{6.103}$$

are frequently encountered. Considering the first of Equation 6.103, we integrate first with respect to r (using the Gaussian technique) to obtain

$$I = \int_{-1}^{1}\int_{-1}^{1} f(r, s)\,\mathrm{d}r\,\mathrm{d}s = \int_{-1}^{1}\sum_{i=1}^{n} [W_i f(r_i, s)]\,\mathrm{d}s = \int_{-1}^{1} g(s)\,\mathrm{d}s \tag{6.104}$$

which, in turn, is integrated via quadrature to obtain

$$I = \sum_{j=1}^{m} W_j g(s_j) \tag{6.105}$$

combining Equations 6.98 and 6.99, we find

$$I = \int_{-1}^{1}\int_{-1}^{1} f(r, s)\,\mathrm{d}r\,\mathrm{d}s = \sum_{j=1}^{m}\sum_{i=1}^{n} W_j W_i f(r_i, s_j) \tag{6.106}$$